$W_n^{(\kappa)}$ algebra associated with the Moyal KdV Hierarchy

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Abstract

We consider the Gelfand-Dickey (GD) structure defined by the Moyal $\star$-product with parameter κ, which not only defines the bi-Hamiltonian structure for the generalized Moyal KdV hierarchy but also provides a $W_n^{(\kappa)}$ algebra containing the Virasoro algebra as a subalgebra with central charge $\kappa^2(n^3-n)/3$. The free-field realization of the $W_n^{(\kappa)}$ algebra is given through the Miura transformation and the cases for $W_3^{(\kappa)}$ and $W_4^{(\kappa)}$ are worked out in detail.

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I. INTRODUCTION

Over the past decade, much attention has been paid to the W-algebars in connection with integrable systems and string theories (see, for example, [1–3] and references therein). For example, it is well-known that the classical realizaion of W_n algebra [4–6] can be constructed from the second Gelfand-Dickey(GD) structure [7,8] associated with the differential Lax operator. Furthermore, the GD bracket with graded commutators was proposed [9,10] for supersymmetric pseudo-differential operators and the classical version of extended supersymmetric W-algebras was obtained.

Recently, the so called dispersionless Lax equations were considered [11–14], which are the quasi-classical limit of the ordinary Lax equations. In such limit, the ordinary (pseudo-) differential Lax operators are replaced by the formal Laurent series in p and the canonical Poisson bracket $\{f(x,p), g(x,p)\} = \partial_p f \partial_x g - \partial_x f \partial_p g$ takes over the role of the commutator $[A, B] = AB - BA$ in the ordinary Lax formalism. Moreover, the quasi-classical limit of the GD structure can be defined and the associated classical limit of W-algebras were constructed [15]. All the mentioned results reveal that the algebraic structures associated with the GD bracket are profound.

More recently, there has been a great deal of interest to study the Moyal deformations of the Lax equations [16–20] in algebraic and/or geometric ways by using the so-called Moyal $\star$-product defined by [21–23] (for a historical survey of $\star$-product, see [24])

$$f \star g = \sum_{s=0}^{\infty} \frac{\kappa^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (\partial_x^j \partial_p^{s-j} f)(\partial_x^{s-j} \partial_p^j g) \tag{1.1}$$

where f and g are two arbitrary functions on the two-dimensional phase space with coordinate (x, p). By (1.1), the Moyal bracket [21–23] is defined by

$$\{f, g\}_\kappa = \frac{f \star g - g \star f}{2\kappa} \tag{1.2}$$

that satisfies the following properties (i) $\{f, g\}_\kappa = -\{g, f\}_\kappa$ (anti-symmetry) (ii) $\{af + bg, h\}_\kappa = a\{f, h\}_\kappa + b\{g, h\}_\kappa$ (linearity) (iii) $\{f, \{g, h\}_\kappa\}_\kappa + \{g, \{h, f\}_\kappa\}_\kappa + \{h, \{f, g\}_\kappa\}_\kappa = 0$ (Jacobi identity). Since the Moyal bracket (1.2) recovers the canonical Poisson bracket in the limit $\kappa \to 0$, i.e. $\lim_{\kappa\to 0}\{f, g\}_\kappa = \partial_p f \partial_x g - \partial_x f \partial_p g$, thus it can be viewed as the higher-order derivative generalization of the canonical Poisson bracket. Motivated by the works described above and consulting the dispersionless KdV hierarchy [13] whose Lax flows are defined by the canonical Poisson bracket, the zero-th order term in κ in (1.2), we attempt to define the the KdV hierarchy by using the Moyal bracket. We shall show that the integrability is still maintained under the Moyal deformation and a W-type algebra emerges naturally from the associated second GD structure.

Our paper is organized as follows: In Sec. II, we recall some basic notions and introduce the Lax formulation of the Moyal KdV hierarchy. In Sec. III, we show that the Gelfand-Dickey structure with respect to the Moyal $\star$-product defines the bi-Hamiltonian structure of the Moyal KdV hierarchy. In Sec. IV, we study the conformal property of the Gelfand-Dickey algebras and explicitly identify the first few primary fields. In Sec. V, by factorizing the Lax operator, the free-field realization for the associated conformal algebras is given. We work out some examples in Sec. VI and present the concluding remarks in Sec. VII.

II. MOYAL KDV HIERARCHY

To begin with, let us consider an algebra of Laurent series of the form $\Lambda = \{A | A = \sum_{i=-\infty}^{N} a_i p^i\}$ with coefficients a_i depending on an infinite set of variables $t_1 \equiv x, t_2, t_3, \cdots$. The algebra Λ can be decomposed into the sub-algebras as $\Lambda = \Lambda_{\geq k} \oplus \Lambda_{<k}, (k = 0, 1, 2.)$ where $\Lambda_{\geq k} = \{A \in \Lambda | A = \sum_{i \geq k} a_i p^i\},\ \Lambda_{<k} = \{A \in \Lambda | A = \sum_{i<k} a_i p^i\}$ and using the notations : $\Lambda_+ = \Lambda_{\geq 0}$ and $\Lambda_- = \Lambda_{<0}$ for short. It's obvious that Λ is an associative but noncommutative algebra under the Moyal $\star$-product. For a given Laurent series A we define its residue as $\mathrm{res}(A) = a_{-1}$ and its trace as $\mathrm{tr}(A) = \int \mathrm{res}(A)$. For any two Laurent series $A = \sum_i a_i p^i$ and $B = \sum_j b_j p^{-j}$ we have

$$\int \mathrm{res}(A \star B) = \int \sum_{i,j} \frac{\kappa^{i-j+1} i!}{(i-j+1)!(j-1)!} (a_i b_j)^{(i-j+1)} = \sum_i \int a_i b_{i+1} \tag{2.1}$$

which is the same as the case in the dispersionless limit $\kappa \to 0$. We shall see that it's because of the nice property (2.1) so that the Hamiltonian formulation for the Moyal KdV becomes possible. Using (2.1) it is easy to show that $\mathrm{tr}\{A, B\}_\kappa = 0$ and $\mathrm{tr}(A \star \{B, C\}_\kappa) = \mathrm{tr}(\{A, B\}_\kappa \star C)$. Here we simply remark that, due to the property (2.1), the Moyal $\star$-product within the trace can be replaced by the ordinary multiplication. However we shall reserve the product for convenience.

Finally, given a functional $F(A) = \int f(a)$ we define its gradient as

$$d_A F = \sum_i \frac{\delta f}{\delta a_i} p^{-i-1}$$

where the variational derivative is defined by

$$\frac{\delta f}{\delta a_k} = \sum_i (-1)^i \left(\partial^i \cdot \frac{\partial f}{\partial a_k^{(i)}} \right),$$

with $a_k^{(i)} \equiv (\partial^i \cdot a_k), \partial \equiv \partial/\partial x$. Note that we shall use the notations $\partial \cdot f = f' = \partial f/\partial x$ and $\partial f = f\partial + f'$ in the following sections.

The Moyal KdV hierarchy is defined by the Lax equations

$$
\begin{aligned}
\frac{\partial L}{\partial t_k} &= \{(L^{1/n})^k_+, L\}_\kappa, \qquad (L^{1/n})^k_+ = (\underbrace{L^{1/n}\star L^{1/n}\star\cdots\star L^{1/n}}_{k})_+ \\
&= \{L, (L^{1/n})^k_-\}_\kappa
\end{aligned} \tag{2.2}
$$

where the Lax operator $L = p^n + \sum_{i=0}^{n-1} u_i p^i$ is a polynomial in p and $L^{1/n} = p + \sum_{i=0} a_i p^{-i}$ is the nth root of L in such a way that

$$
L = \underbrace{L^{1/n}\star L^{1/n}\star\cdots\star L^{1/n}}_{n}.
$$

From the definition of the Moyal bracket, the highest order in p on the right-hand side of the Lax equations (2.2) is $n-2$. That means u_{n-1} is trivial in evolution equations and thus can be dropped in the Lax formulation. However, this is not the case for the Hamiltonian formulation (see next section).

Let us work out the simplest example. For $n = 2$, $L = p^2 + u$ and we have $L^{1/2} = p + \sum_{i=1}^{\infty} a_i p^{-i}$ with

$$
\begin{aligned}
a_1 &= \frac{1}{2}u, \\
a_3 &= -\frac{1}{8}u^2, \\
a_5 &= \frac{1}{16}u^3 + \frac{1}{8}\kappa^2(u_x^2 - 2uu_{xx}), \\
a_7 &= -\frac{5}{128}u^4 - \frac{5}{16}\kappa^2(uu_x^2 - 2u^2u_{xx}) - \frac{1}{8}\kappa^4(u_{xx}^2 - 2u_xu_{xxx} + 2uu^{(4)}), \\
a_9 &= \frac{7}{256}u^5 + \frac{35}{64}\kappa^2(u^2u_x^2 - 2u^3u_{xx}) + \frac{7}{16}\kappa^4(uu_{xx}^2 - 4uu_xu_{xxx} + 3u^2u^{(4)} - 3u_{xx}u_x^2) \\
&\quad + \frac{1}{4}\kappa^6(u_xu^{(5)} - uu^{(6)}),
\end{aligned}
$$

and $a_{2k} = 0$ etc. The first few Lax flows are given by

$$
\begin{aligned}
u_{t_1} &= u_x, \\
u_{t_3} &= \frac{3}{2}uu_x + \kappa^2 u_{xxx}, \\
u_{t_5} &= \frac{15}{8}u^2u_x + \frac{5}{2}\kappa^2(uu_{xxx} + 2u_xu_{xx}) + \kappa^4u^{(5)}, \\
&\cdots
\end{aligned} \tag{2.3}
$$

The set of equations (2.3) form what we call the Moyal KdV hierarchy which can also be obtained from the reduction of the Moyal KP hierarchy [18] or noncommutative zero-curvature equations [20]. Note that, when $\kappa = 0$, all higher-order derivative terms disappear and the Moyal KdV hierarchy reduces to the dispersionless KdV hierarchy which is of the hydrodynamic type [25]. In this sense, the Moyal parameter κ characterizes the dispersion effect. On the other hand, when $\kappa = 1/2$, the Moyal

KdV hierarchy (2.3) recovers the ordinary KdV hierarchy. This is not an accident due to the fact that, at $\kappa = 1/2$, the Moyal KP hierarchy is isomorphic with the ordinary KP hierarchy [18,19]. Thus the Moyal KdV is isomorphic with the ordinary KdV as well. For example, it is not hard to show that the Moyal Boussinesq hierarchy ($L = p^3 + u_1 p + u_0$) at $\kappa = 1/2$ is isomorphic with the ordinary Boussinesq hierarchy ($L = \partial^3 + v_1 \partial + v_0$) by identifying

$$u_1 = v_1, \qquad u_0 = v_0 - \frac{1}{2} v_1'. \tag{2.4}$$

Finally we like to remark that a similar construction was established in [16]. However, it was pointed out in [18] that the Lax equations obtained there do not have dispersionless limit as $\kappa \to 0$, not without a scaling transformation.

III. BI-HAMILTONIAN STRUCTURE

Having introduced the Lax formalism, in this section, we would like to discuss the Hamiltonian formalism of the Moyal KdV hierarchy. For the Lax operator $L = p^n + \sum_{i=0}^{n-1} u_i p^i$ and functionals $F[L]$ and $G[L]$ we define the second Gelfand-Dickey bracket [8] with respect to the Moyal $\star$-product as

$$\{F, G\}_2 = \mathrm{tr}(J^{(2)}(d_L F) \star d_L G) = \int \mathrm{res}(J^{(2)}(d_L F) \star d_L G) \tag{3.1}$$

in which $J^{(2)}$ is the Adler map defined by [26]

$$\begin{aligned} J^{(2)}(X) &= \{L, X\}_{\kappa +} \star L - \{L, (X \star L)_+\}_\kappa, \\ &= \{L, (X \star L)_-\}_\kappa - \{L, X\}_{\kappa -} \star L \end{aligned} \tag{3.2}$$

where $X = \sum_{i=1}^{n} x_i p^{-i-1}$. To verify that $J^{(2)}$ is indeed Hamiltonian, we have to check that the Poisson bracket defined in (3.1) is antisymmetric and obeys the Jacobi identity. For antisymmetry, it can be easily shown that $\{F, G\}_2 = -\{G, F\}_2$ by using the cyclic property of the trace. For the Jacobi identity, instead of direct computation, we shall justify it by the Kupershmidt-Wilson (KW) theorem [27] that will be done in Sec.V.

Since $J^{(2)}(X)$ is linear in X and has order at most $n-1$ thus

$$J^{(2)}(X) = \sum_{i,j=0}^{n-1} (J^{(2)}_{ij} \cdot x_j) p^i$$

where $J^{(2)}_{ij}$ are differential operators.

To impose the reduction $u_{n-1} = 0$, the standard Dirac procedure [6] gives

$$\hat{J}^{(2)}(X) = \{L, X\}_{\kappa +} \star L - \{L, (X \star L)_+\}_\kappa + \frac{1}{n}\{L, \int^x \mathrm{res}\{L, X\}_\kappa\}_\kappa \tag{3.3}$$

or, in components,

$$\hat{J}^{(2)}_{ij} = J^{(2)}_{ij} - J^{(2)}_{i,n-1}(J^{(2)}_{n-1,n-1})^{-1}J^{(2)}_{n-1,j}. \tag{3.4}$$

Hence the reduced Poisson brackets for u_i can be expressed as

$$\{u_i(x), u_j(y)\}^D_2 = \hat{J}^{(2)}_{ij} \cdot \delta(x-y). \tag{3.5}$$

So far we only discuss the second Poisson structure. To obtain the first structure, as usual, one can deform the second structure by shifting $L \to L + \lambda$ (or $u_0 \to u_0 + \lambda$) and extract the first structure from the term proportional to λ. It turns out that

$$J^{(1)}(X) = \{L, X\}_{\kappa+} = \sum_{i=0}^{n-2}(J^{(1)}_{ij} \cdot x_j)p^i$$

which is compatible with the reduction $u_{n-1} = 0$. The first Poisson brackets for u_i read

$$\{u_i(x), u_j(y)\}_1 = J^{(1)}_{ij} \cdot \delta(x-y). \tag{3.6}$$

Using the GD brackets (3.5) and (3.6), the Lax flows (2.2) can be written as Hamiltonian flows as

$$\frac{\partial L}{\partial t_k} = \{H_k, L\}^D_2 = \{H_{k+n}, L\}_1$$

or, in components,

$$\frac{\partial u_i}{\partial t_k} = \hat{J}^{(2)}_{ij} \cdot \frac{\delta H_k}{\delta u_j} = J^{(1)}_{ij} \cdot \frac{\delta H_{k+n}}{\delta u_j}$$

where the Hamiltonians H_k are defined by

$$H_k = \frac{n}{k}\int \mathrm{res}(L^{1/n} \star \cdots \star L^{1/n}). \tag{3.7}$$

For $n = 2$, $L = p^2 + u$ and the bi-Hamiltonian structure is given by

$$\begin{aligned} \{u(x), u(y)\}_1 &= 2\partial \cdot \delta(x-y), \\ \{u(x), u(y)\}^D_2 &= [2\kappa^2\partial^3 + 2u\partial + u_x] \cdot \delta(x-y). \end{aligned} \tag{3.8}$$

The first few Hamiltonians from (3.7) are

$$\begin{aligned} H_1 &= \int u, \\ H_3 &= \frac{1}{4}\int u^2, \\ H_5 &= \frac{1}{8}\int (u^3 + 2\kappa^2 u u_{xx}), \\ H_7 &= \frac{1}{64}\int (5u^4 - 40\kappa^2 u u_x^2 + 16\kappa^4 u_{xx}^2), \end{aligned}$$

which together with (3.8) implies

$$\frac{\partial u}{\partial t_{2n+1}} = [2\kappa^2\partial^3 + 2u\partial + u_x] \cdot \frac{\delta H_{2n+1}}{\delta u} = 2\partial \cdot \frac{\delta H_{2n+3}}{\delta u}.$$

Therefore we obtain the following recursion relation

$$\frac{\partial u}{\partial t_{2n+1}} = R \cdot \frac{\partial u}{\partial t_{2n-1}}$$

with the recursion operator [20]

$$R = \hat{J}^{(2)}(J^{(1)})^{-1} = \kappa^2\partial + u + \frac{1}{2}u_x\partial^{-1}$$

where the inverse operator ∂^{-1} is realized as $\partial^{-1} \cdot f = \int^x f$.

IV. MOYAL DEFORMATION OF CLASSICAL W_N-ALGEBRAS

In general, for the nth-order generalized Moyal KdV, we can substitute L and X into $J^{(1)}(X)$ and $\hat{J}^{(2)}(X)$ to read off the Hamiltonian operators $J^{(1)}_{ij}$ and $\hat{J}^{(2)}_{ij}$. For the first structure, it is quite easy to show

$$\begin{aligned}
J^{(1)}_{ij} &= n\partial, \qquad (i+j=n-2)\\
J^{(1)}_{ij} &= \kappa^{<n-j-i-2>}\binom{n}{<n-j-i-2>+1}\partial^{<n-j-i-2>+1}\\
&\quad + \sum_{l=i}^{n-4-j}\sum_{m=0}^{<l-i>+1}\kappa^{<l-i>}\binom{l+j+2}{<l-i>+1-m}\binom{j+m}{j}u^{(m)}_{l+j+2}\partial^{<l-i>+1-m},\\
&\quad (i+j\le n-4)
\end{aligned}$$

and $J^{(1)}_{ij}=0$ otherwise, where $<..>=0$ unless it's an even number. On the other hand, the case for the second structure is more complicated. From (3.2) we have

$$\begin{aligned}
J^{(2)}(X) = &\sum_{l=0}^{n-1}\sum_{m=0}^{n-l-1}\sum_{k=0}^{n}\sum_{s=0}^{m+k}\sum_{i=0}^{m+k-s}\sum_{q=m}^{n-l-1}\sum_{j=0}^{q-m}\kappa^{<k+q-s-1>}(-1)^i\binom{m}{s-k+i}\binom{k}{i}\\
&\binom{q+l+1}{j}\binom{q+l-m-j}{l}u_k^{(m+k-s-i)}\left(u^{(q-m-j)}_{q+l+1}x^{(j)}_l\right)^{(i)}p^s.
\end{aligned}$$

In particular,

$$\begin{aligned}
J^{(2)}_{n-1,n-1} &= -n\partial,\\
J^{(2)}_{s,n-1} &= \sum_{k=s}^{n}\kappa^{<k-s-1>}(-1)^{k-s}\binom{k}{s}u_k\partial^{k-s},
\end{aligned}$$

$$
\begin{aligned}
J^{(2)}_{s,n-2} &= \sum_{k=s}^{n}\sum_{q=0}^{1}\sum_{j=0}^{q}\kappa^{<k-s+q-1>}(-1)^{k-s}\binom{k}{s}\binom{q+n-1}{j}\binom{q-j+n-2}{n-2}u_k\partial^{k-s}u^{(q-j)}_{q+n-1}\partial^j\\
&+\sum_{k=s}^{n}\sum_{i=0}^{1+k-s}\kappa^{<k-s>}(-1)^i\binom{1}{1+k-s-i}\binom{k}{i}u_k^{1+k-s-i}\partial^i
\end{aligned}
$$

which together with (3.4) yields the reduced Poisson algebras

$$
\begin{aligned}
\{u_{n-2}(x),u_{n-2}(y)\}^D_2 &= [\kappa^2\frac{(n^3-n)}{3}\partial^3+2u_{n-2}\partial+u'_{n-2}]\cdot\delta(x-y),\\
\{u_{n-3}(x),u_{n-2}(y)\}^D_2 &= [3u_{n-3}\partial+u'_{n-3}]\cdot\delta(x-y),\\
\{u_{n-4}(x),u_{n-2}(y)\}^D_2 &= \Bigg[\kappa^4\frac{(n+1)n(n-1)(n-2)(n-3)}{30}\partial^5\\
&+\kappa^2\left(\frac{(n-2)(n-3)(n+2)}{3}u_{n-2}\partial^3+\frac{(n-2)(n-3)}{2}u'_{n-2}\partial^2\right)\\
&+4u_{n-4}\partial+u'_{n-4}\Bigg]\cdot\delta(x-y),\\
\{u_{n-3}(x),u_{n-3}(y)\}^D_2 &= \Bigg[-\kappa^4\frac{(n+2)(n+1)n(n-1)(n-2)}{45}\partial^5-\kappa^2\Bigg(\frac{2(n-2)(n+2)}{3}u_{n-2}\partial^3\\
&+(n-2)(n+2)u'_{n-2}\partial^2+n(n-2)u''_{n-2}\partial+\frac{(n-1)(n-2)}{3}u'''_{n-2}\Bigg)\\
&-\frac{2(n-2)}{n}u_{n-2}\partial u_{n-2}+4u_{n-4}\partial+2u'_{n-4}\Bigg]\cdot\delta(x-y).
\end{aligned}
\tag{4.1}
$$

Thus $u_{n-2}(x)$ can be identified as the energy-momentum $T(x)$ of the Virasoro algebra with central charge $c(n,\kappa)=\kappa^2(n^3-n)/3$ whereas the other coefficient functions u_i, with respect to u_{n-2}, are not primaries except u_{n-3} which is a spin-3 primary field W_3. However, if we appropriately linearly combine the differential polynomials of u_i, then the primary fields $W_i (i\geq 4)$ can be constructed such that $\{W_i(x),u_{n-2}(y)\}^D_2 = [iW_i\partial+W'_i]\cdot\delta(x-y)$. In summary, we have

$$
\begin{aligned}
T &= u_{n-2},\\
W_3 &= u_{n-3},\\
W_4 &= u_{n-4}-\kappa^2\frac{(n-2)(n-3)}{10}u''_{n-2}-\kappa^4\frac{(n-2)(n-3)(5n+7)}{10(n^3-n)}u^2_{n-2}
\end{aligned}
\tag{4.2}
$$

etc. Based on the above discussions and some computations we expect that the $W_n^{(1/2)}$ algebra coincides with the classical W_n algebra associated with the ordinary differential Lax operator $L=\partial^n+\sum_{u=0}^{n-2}v_i\partial^i$. We remark that the above relationship between the primaries W_i and the coefficient functions u_i is different from those tabulated in [6] due to the fact that the Moyal KdV at $\kappa=1/2$ is isomorphic with the ordinary KdV. As an illustration, again, let us consider the differential Lax operator $L=\partial^3+v_1\partial+v_0$ $(n=3)$. The primary fields are given by [5]

$$T = v_1, \qquad W_3 = v_0 - \frac{1}{2}v_1'$$

which together with the isomorphism (2.4) implies

$$T = u_1, \qquad W_3 = u_0.$$

as desired. For higher primaries, such as W_4 in (4.2), one can verify the corresponding modifications in the same manner.

Before ending this section, it should be mentioned that, in the limit $\kappa \to 0$, both algebraic structures $J_{ij}^{(1)}$ and $\hat{J}_{ij}^{(2)}$ will reduce to their corresponding dispersionless counterparts [15]. Particularly, for the second structure $\hat{J}_{ij}^{(2)}$, the coefficient functions u_i under this limit are already primary fields with respect to u_{n-2}, the generator of DiffS^1.

V. KW THEOREM AND FREE-FIELD REALIZATION

Recall that the Poisson bracket defined by the Adler map (3.2) is given by

$$\{F, G\}_2(L) = \mathrm{tr}\left[J^{(2)}(d_L F) \star d_L G\right]$$

where F and G are functionals of L. Suppose the Lax operator L can be factorized as $L = L_1 \star L_2$, then from the variation

$$\begin{aligned} \delta F &= \mathrm{tr}(d_L F \delta L) = \mathrm{tr}(d_{L_1} F \delta L_1 + d_{L_2} F \delta L_2), \\ &= \mathrm{tr}(d_L F \delta L_1 \star L_2 + d_L F L_1 \star \delta L_2) \end{aligned}$$

and (2.1) we have

$$d_{L_1} F = L_2 d_L F, \qquad d_{L_2} F = d_L F L_1. \tag{5.1}$$

With the use of (2.1) and (5.1) the proof for the KW theorem is almost a repetition of that for the dispersionless case [28]. We shall not spell it out here. It turns out that

$$\{F, G\}_2(L_1 \star L_2) = \{F, G\}_2(L_1) + \{F, G\}_2(L_2).$$

Hence, if we factorize the Lax operator as $L = p^n + \sum_{i=1}^n u_i p^i = (p+\phi_1)\star\cdots\star(p+\phi_n)$ which defines a Miura transformation between the coefficients $\{u_i\}$ and the Miura variables $\{\phi_i\}$, then the second GD bracket becomes

$$\{F, G\}_2(L) = \sum_{i=1}^n \int (\frac{\delta F}{\delta \phi_i})'(\frac{\delta G}{\delta \phi_i})$$

which yields the Poisson brackets

$$\{\phi_i(x), \phi_j(y)\}_2 = -\delta_{ij}\partial \cdot \delta(x-y), \qquad i,j = 1, \cdots, n$$

The above brackets immediately justify the Jacobi identity for the second GD bracket (3.1) as we claimed in Sec. III. To impose the reduction $u_{n-1} = 0$, in view of $u_{n-1} = \sum_{i=1}^{n} \phi_i$, we have

$$\{\phi_i(x), \phi_j(y)\}_2^D = \left[\frac{1}{n} - \delta_{ij}\right] \partial \cdot \delta(x-y). \tag{5.2}$$

Equation (5.2) enables us to write down the reduced Poisson brackets of u_i through the Miura transformation. Since the Poisson matrix in (5.2) is a $n \times n$ symmetric matrix, thus it can be diagonalized to obtain the free-field realization of the $W_n^{(\kappa)}$ algebra.

VI. $W_3^{(\kappa)}$ AND $W_4^{(\kappa)}$-ALGEBRAS

Let us work out the examples $n = 3$ and $n = 4$. For $n = 3$, $L = p^3 + u_1 p + u_0$ and $X = x_1 p^{-2} + x_0 p^{-1}$ then the Poisson algebras for $u_1 = T$ and $u_0 = W_3$ are

$$\begin{aligned}
\{T(x), T(y)\}_2^D &= [8\kappa^2\partial^3 + 2T\partial + T'] \cdot \delta(x-y), \\
\{W_3(x), T(y)\}_2^D &= [3W_3\partial + W_3'] \cdot \delta(x-y), \\
\{W_3(x), W_3(y)\}_2^D &= [-\frac{8}{3}\kappa^4\partial^5 - \frac{1}{3}\kappa^2(10T\partial^3 + 15T'\partial^2 + 9T''\partial + 2T''') - \frac{2}{3}T\partial T] \cdot \delta(x-y)
\end{aligned}$$

which is $W_3^{(\kappa)}$. In the limit $\kappa \to 0$, the $W_3^{(0)}$ algebra is just w_3 [15] which is a nonlinear extension of DiffS^1 by a spin-3 primary field, while $\kappa = 1/2$, $W_3^{(1/2)}$ is nothing but the classical realization of the Zamolodchikov's W_3 algebra presented in [5].

If we factorize the Lax operator as

$$L = (p + \phi_1) \star (p + \phi_2) \star (p - \phi_1 - \phi_2)$$

then the primary fields can be expressed in terms of the Miura variables as

$$\begin{aligned}
T &= -(\phi_1^2 + \phi_1\phi_2 + \phi_2^2) - 2\kappa(2\phi_1' + \phi_2'), \\
W_3 &= -\phi_1\phi_2(\phi_1 + \phi_2) - \kappa(\phi_2\phi_1' + 2\phi_2\phi_2' + 3\phi_1\phi_2') - 2\kappa^2\phi_2''.
\end{aligned}$$

Through the above Miura transformation the $W_3^{(\kappa)}$ can be rederived by using those brackets of ϕ_i.

For $n = 4$, $L = p^4 + u_2 p^2 + u_1 p + u_0$ and $X = x_2 p^{-3} + x_1 p^{-2} + x_0 p^{-1}$. Then from (3.3) we have

$$
\begin{aligned}
\{u_2(x), u_2(y)\}_2^D &= [20\kappa^2\partial^3 + 2u_2\partial + u_2'] \cdot \delta(x-y),\\
\{u_1(x), u_2(y)\}_2^D &= [3u_1\partial + u_1'] \cdot \delta(x-y),\\
\{u_0(x), u_2(y)\}_2^D &= [4\kappa^4\partial^5 + \kappa^2(4u_2\partial^3 + u_2'\partial^2) + 4u_0\partial + u_0'] \cdot \delta(x-y),\\
\{u_1(x), u_1(y)\}_2^D &= [-16\kappa^4\partial^5 - 2\kappa^2(4u_2\partial^3 + 6u_2'\partial^2 + 4u_2''\partial + u_2''')\\
&\quad -u_2\partial u_2 + 4u_0\partial + 2u_0'] \cdot \delta(x-y),\\
\{u_0(x), u_1(y)\}_2^D &= \left[-\kappa^2(5u_1\partial^3 + 4u_1'\partial^2 + u_1''\partial) - \frac{1}{2}u_1\partial u_2\right] \cdot \delta(x-y),\\
\{u_0(x), u_0(y)\}_2^D &= [4\kappa^6\partial^7 + \kappa^4(6u_2\partial^5 + 15u_2'\partial^4 + 20u_2''\partial^3 + 15u_2'''\partial^2 + 6u_2^{(4)}\partial + u_2^{(5)})\\
&\quad +\kappa^2((4u_0 + 2u_2^2)\partial^3 + (6u_0' + 6u_2u_2')\partial^2 + (4u_2u_2'' + 2(u_2')^2 + 4u_0'')\partial\\
&\quad +(u_2u_2''' + u_2'u_2'' + u_0''')) + (2u_0u_2 - 3u_1^2/4)\partial\\
&\quad +(u_2u_0' + u_0u_2' - 3u_1u_1'/4)] \cdot \delta(x-y).
\end{aligned}
$$

According to (4.2), if we define $T = u_2, W_3 = u_1$, and $W_4 = u_0 - \kappa^2 u_2''/5 - 9\kappa^4 u_2^2/100$ then the brackets involving W_4 are

$$
\begin{aligned}
\{W_4(x), T(y)\}_2^D &= [4W_4\partial + W_4'] \cdot \delta(x-y),\\
\{W_4(x), W_3(y)\}_2^D &= \Big[-\frac{9}{50}\kappa^4(3TW_3\partial + 2TW_3') - \frac{2}{5}\kappa^2(14W_3\partial^3 + 14W_3'\partial^2 + 6W_3''\partial + W_3''')\\
&\quad -\frac{1}{2}(W_3T\partial + W_3T')\Big] \cdot \delta(x-y),\\
\{W_4(x), W_4(y)\}_2^D &= \Big[\frac{\kappa^{10}}{125}(81T^2\partial^3 + 243TT'\partial^2 + 243TT''\partial + 81TT''')\\
&\quad -\frac{\kappa^8}{2500}(243T^2T' + 81T^3\partial) + \frac{\kappa^6}{250}(800\partial^7 - 288T^2\partial^3 - 864TT'\partial^2\\
&\quad +(54(T')^2 - 918TT'')\partial + 27T'T'' - 315TT''')\\
&\quad +\frac{\kappa^4}{100}(448T\partial^5 + 1120T'\partial^4 + 1344T''\partial^3 + 896T'''\partial^2\\
&\quad +(18T^3 + 120T^{(4)} - 144TW_4)\partial + 27T^2T' - 72(TW_4)' + 48T^{(5)})\\
&\quad +\frac{\kappa^2}{5}((12W_4 + 10T^2)\partial^3 + (18W_4' + 30TT')\partial^2\\
&\quad +(10W_4'' + 22TT'' + 10(T')^2)\partial + 2W_4''' + 6T'T'' + 6TT''')\\
&\quad +T\partial W_4 + W_4\partial T - \frac{3}{4}W_3\partial W_3\Big] \cdot \delta(x-y)
\end{aligned}
$$

which together with the rest of the bracket constitute the $W_4^{(\kappa)}$. Since the discussions for the limiting cases $W_4^{(0)}$ and $W_4^{(1/2)}$ are similar to those of $W_3^{(\kappa)}$, hence we omit it here.

VII. CONCLUDING REMARKS

We have studied the integrability of the Moyal KdV hierarchy from Lax and Hamiltonian point of views. We show that the bi-Hamiltonian structure is encoded by the GD structure defined by the Moyal $\star$-product. We work out the GD brackets and investigate the associated conformal algebras. The $W_n^{(\kappa)}$ algebra we obtained can be viewed as an one parameter deformation of the classical W_n algebra arising from the differential Lax formalism. We prove the KW theorem for the second GD structure by factorizing the Lax operator and obtain a κ-independent free-field realization.

In spite of the results obtained, few remarks are in order. First, we may generalize the construction in this paper to the Moyal KP hierarchy [18] and investigate its associated $W_{KP}^{(\kappa)}$ algebra by considering the Lax operator $L = p^n + \sum_{i=1}^{\infty} u_{n-i} p^{n-i}$. We expect that $W_{KP}^{(1/2)}$ would coincide with the W_{KP} algebra [29] defined by the pseudo-differential operator. Secondly, it would be interesting to search for some topological models so that their RG flows in module spaces are governed by the Moyal KdV equations. Works in these directions are now in progress.

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Note added: After submission of this manuscript for publication we become aware of the preprint by A. Das and Z. Popowicz [30] which partly overlaps our work.

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